

# ON A FAMILY OF LAGRANGIAN SUBMANIFOLDS IN BIDISKS AND LAGRANGIAN HOFER METRIC

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**ABSTRACT.** We construct a family of uncountably many Lagrangian submanifolds in the standard bidisks such that the Lagrangian Hofer diameter associated to each Lagrangian submanifold is unbounded. We also prove a certain inequality of the Lagrangian Hofer metric which is of the same type as S. Seyfaddini's for the case of the real form of the complex  $n$ -ball.

## 1. INTRODUCTION

For a symplectic manifold  $(M, \omega)$ , we denote by  $\text{Ham}_c(M, \omega)$  the group of all compactly supported Hamiltonian diffeomorphisms on  $(M, \omega)$ . For a Lagrangian submanifold  $L$  of  $(M, \omega)$ ,  $\mathcal{L}(L)$  denotes the set of Lagrangian submanifolds which are Hamiltonian isotopic to  $L$ . The *Lagrangian Hofer pseudo-metric*  $d$  on  $\mathcal{L}(L)$  is defined by using the *Hofer norm*  $\|\cdot\|$ , which is introduced in [Ho90], as follows.

$$d(L_0, L_1) := \inf \{ \|\phi\| \mid \phi(L_0) = L_1, \phi \in \text{Ham}_c(M, \omega) \}.$$

The Hofer norm  $\|\phi\|$  is defined by

$$\|\phi\| := \inf \int_0^1 \left( \max_{p \in M} H(t, p) - \min_{p \in M} H(t, p) \right) dt,$$

where the infimum runs over all compactly supported Hamiltonians  $H \in C_c^\infty([0, 1] \times M)$  having time-one map  $\phi_H^1$  equal to  $\phi$ .

Chekanov showed in [Ch00] that this pseudo-metric  $d$  is non-degenerate for any closed and connected Lagrangian submanifolds in tame symplectic manifolds. Although our Lagrangian submanifolds are not closed, the same proof as Chekanov's yields that  $d$  is also non-degenerate for our cases below.

In [Kh09], Khanevsky proved unboundedness of this metric when the ambient space  $M$  is an open unit disk  $B^2 := \{z \in \mathbb{C} \mid |z| < 1\} \subset \mathbb{C}$  and the Lagrangian submanifold  $L$  is the real form  $Re(B^2) := \{z \in B^2 \mid \text{Im } z = 0\}$  of the open unit disk. Seyfaddini generalized Khanevsky's unboundedness result to the case of higher dimensional open unit ball  $B^{2n}$  in [Se14].

In this paper, by adopting Seyfaddini's technique, we prove unboundedness of metric spaces  $\mathcal{L}(L)$  for a certain continuous family of non-compact Lagrangian submanifolds in bi-disks, which are mutually non-Hamiltonian isotopic.

**1.1. Main Result.** Let  $B^2(r) \subset \mathbb{C}$  be the open disk of radius  $r > 0$  equipped with a symplectic structure  $2\omega_0$ , where  $\omega_0$  is the standard symplectic structure on  $\mathbb{C}$  so that  $\text{vol}(D(r)) = 2\pi r^2$ . We denote by  $B^2$  the open unit disk  $B^2(1)$ . We put  $(B^2 \times B^2, \bar{\omega}_0) := (B^2(1) \times B^2(1), 2\omega_0 \oplus 2\omega_0)$  and define Lagrangian submanifolds  $L_\delta$  by

$$L_\delta := T_\delta \times \text{Re}(B^2) \subset B^2 \times B^2$$

for each  $1/2 < \delta \leq 1$ . Here

$$T_\delta := \{|z_1|^2 = 1/(2\delta)\} \subset B^2$$

and  $\text{Re}(B^2)$  is the real form of  $B^2$ .

We study the Lagrangian Hofer metric spaces  $(\mathcal{L}(L_\delta), d)$  in this paper. We obtain the following:

**Theorem 1.1.** *For any  $1/2 < \delta \leq 1$ ,  $(\mathcal{L}(L_\delta), d)$  has an infinite diameter.*

In addition to unboundedness, we prove the following inequality for a subfamily of  $\{L_\delta\}$ .

**Theorem 1.2.** *For any  $(2 + \sqrt{3})/4 < \delta \leq 1$ , there exists a map  $\Phi_\delta : C_c^\infty((0, 1)) \rightarrow \mathcal{L}(L_\delta)$  such that*

$$\frac{\|f - g\|_\infty - D_\delta}{C_\delta} \leq d(\Phi_\delta(f), \Phi_\delta(g)) \leq \|f - g\|,$$

where  $C_\delta$  and  $D_\delta$  denote positive constants.

In this statement,  $C_c^\infty((0, 1))$  denotes the space of compactly supported smooth functions on an open interval  $(0, 1)$  and the two norms on  $C_c^\infty((0, 1))$  is defined by

$$\|f\|_\infty := \max_{x \in (0, 1)} |f(x)|,$$

and

$$\|f\| := \max_{x \in (0, 1)} f(x) - \min_{x \in (0, 1)} f(x).$$

These norms are equivalent. We note that  $\|f\|_\infty = \|f\|$  for any non-negative functions  $f \geq 0$ .

**Remark 1.1.** (1) In [Se14], Seyfaddini proved the same type inequality as in Theorem 1.2 for the case of the real form  $\text{Re}(B^{2n})$ . To prove the inequality, he used a family of quasi-morphisms on  $\text{Ham}_c(B^{2n})$  which were constructed as pullbacks of the *single* Calabi quasi-morphism on  $\text{Ham}_c(\mathbb{C}P^n)$  in [EP03] via the same family of conformally symplectic embeddings in [BEP04].  
(2) On the other hand, to prove Theorem 1.2, we use pullbacks of the *family* of Calabi quasi-morphisms on  $\text{Ham}_c(S^2 \times S^2)$  constructed by Fukaya-Oh-Ohta-Ono in [FOOO11b].  
(3) As for the condition on  $\delta$  in Theorem 1.2, see Remark 4.1.

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## 2. CALABI QUASI-MORPHISMS ON $\text{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$

In [BEP04], Biran-Entov-Polterovich used a family of conformally symplectic embeddings to obtain a continuum of linearly independent Calabi quasi-morphisms on  $\text{Ham}_c(B^n, \omega_0)$  as their pullbacks of a quasi-morphism on  $\text{Ham}(\mathbb{C}P^n, \omega_{FS})$ . In [Se14], Seyfaddini used the same family of conformally symplectic embeddings and constructed a family of quasi-morphisms on  $\text{Ham}_c(B^{2n})$  to prove unboundedness of  $\mathcal{L}(Re(B^{2n}), d)$ .

In this section, we also construct quasi-morphisms on  $\text{Ham}_c(B^2 \times B^2)$  associated with Fukaya-Oh-Ohta-Ono's symplectic quasi-morphisms  $\mu_{e_\tau}^{\mathfrak{b}(\tau)}$  as in [Se14].

**2.1. Calabi quasi-morphisms and symplectic quasi-states.** Entov and Polterovich developed a way to construct *Calabi quasi-morphisms* and *symplectic quasi-states* for some closed symplectic manifold  $(M, \omega)$  in a series of papers [EP03, EP06, EP09]. In this section, we briefly recall several terminologies and a generalization of their construction.

A *quasi-morphism* on a group  $G$  is a function  $\mu : G \rightarrow \mathbb{R}$  which satisfies the following property: there exists a constant  $D \geq 0$  such that

$$|\mu(g_1g_2) - \mu(g_1) - \mu(g_2)| \leq D \text{ for all } g_1, g_2 \in G.$$

The smallest number of such  $D$  is called the *defect* of  $\mu$  and we denote by  $D_\mu$ . A quasi-morphism  $\mu$  is called *homogeneous* if  $\mu(g^m) = m\mu(g)$  for all  $m \in \mathbb{Z}$ .

For any proper open subset  $U \subset M$ , the subgroup  $\text{Ham}_U(M, \omega)$  is defined as the set which consists of all elements  $\phi \in \text{Ham}(M, \omega)$  generated by a time-dependent Hamiltonian  $H_t \in C^\infty(M)$  supported in  $U$ . We denote by  $\widetilde{\text{Ham}}_U(M, \omega)$  the universal covering space of  $\text{Ham}_U(M, \omega)$ . The Calabi morphism  $\widetilde{\text{Cal}}_U : \widetilde{\text{Ham}}_U(M^{2n}, \omega) \rightarrow \mathbb{R}$  is defined by

$$\widetilde{\text{Cal}}_U(\tilde{\phi}_H) := \int_0^1 dt \int_M H_t \omega^n,$$

where  $\phi_H^1 \in \text{Ham}_U(M, \omega)$  and  $\tilde{\phi}_H$  is the homotopy class of the Hamiltonian path  $\{\phi_H^t\}_{t \in [0,1]}$  with fixed endpoints. If  $\omega$  is exact on  $U$ ,  $\widetilde{\text{Cal}}_U$  descends to  $\text{Cal}_U : \text{Ham}_U(M, \omega) \rightarrow \mathbb{R}$ .

A subset  $X \subset M$  is called *displaceable* if there exists a  $\phi \in \text{Ham}(M, \omega)$  such that  $\phi(X) \cap \overline{X} = \emptyset$ .

**Definition 2.1** ([EP03]). A function  $\mu : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$  is called a homogeneous Calabi quasi-morphism if  $\mu$  is homogeneous quasi-morphism and satisfies

- (Calabi property) If  $\tilde{\phi} \in \widetilde{\text{Ham}}_U(M, \omega)$  and  $U$  is a displaceable open subset of  $M$ , then

$$(2.1) \quad \mu(\tilde{\phi}) = \widetilde{\text{Cal}}_U(\tilde{\phi}),$$

where we regard  $\tilde{\phi}$  as an element in  $\widetilde{\text{Ham}}(M, \omega)$ .

For each non-zero element of quantum (co)homology  $a \in QH(M)$ , the *spectral invariant*  $\rho(\cdot; a) : C^\infty([0, 1] \times M) \rightarrow \mathbb{R}$  is defined in terms of Hamiltonian Floer theory (see [Oh97], [Sc00], [Vi92] for the earlier constructions and [Oh05] for the general non-exact case).

In [FOOO11b], Fukaya-Oh-Ohta-Ono deformed spectral invariants and obtained  $\rho^b(\cdot; a)$  by using an even degree cocycle  $b \in H^{even}(M, \Lambda_0)$ , where  $a$  is an element of *bulk-deformed quantum cohomology*  $QH_b(M, \Lambda)$  (see also [Us11] for a similar deformation of spectral invariants). Here coefficient ring  $\Lambda_0$ , which is called *universal Novikov ring*, and its quotient field  $\Lambda$  are defined by

$$\begin{aligned} \Lambda_0 &:= \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i \rightarrow \infty} \lambda_i = +\infty \right\}, \\ \Lambda &:= \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i \rightarrow \infty} \lambda_i = +\infty \right\} \cong \Lambda_0[T^{-1}]. \end{aligned}$$

Every element  $\tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$  is generated by some time-dependent Hamiltonian  $H$  which is *normalized* in the sense  $\int_M H_t \omega^n = 0$  for any  $t \in [0, 1]$ . The spectral invariant  $\rho^b(\cdot; a)$  has the homotopy invariance property: if  $F, G$  are normalized Hamiltonians and  $\tilde{\phi}_F = \tilde{\phi}_G$ , then  $\rho^b(F; a) = \rho^b(G; a)$  (see Theorem 7.7 in [FOOO11b]). Hence, the spectral invariant descends to  $\rho^b(\cdot; a) : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$  as follows:

$$\rho^b(\tilde{\phi}_H; a) := \rho^b(\underline{H}; a) \quad \text{for any } H \in C^\infty([0, 1] \times M),$$

where we denote by  $\underline{H}$  the normalization of  $H$ :

$$\underline{H}_t := H_t - \frac{1}{\text{vol}(M)} \int_{M^{2n}} H_t \omega^n, \quad \text{vol}(M) := \int_{M^{2n}} \omega^n.$$

By using this (bulk-deformed) spectral invariant  $\rho^b(\cdot; a)$ , as in a series of papers [EP03, EP06, EP09], they constructed a function  $\mu_e^b : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$  by

$$\mu_e^b(\tilde{\phi}) := \text{vol}(M) \lim_{m \rightarrow +\infty} \frac{\rho^b(\tilde{\phi}^m; e)}{m},$$

where  $e \in QH_b(M, \Lambda)$  is an idempotent.

The following theorem is the generalization of Theorem 3.1 in [EP03].

**Theorem 2.1** (Theorem 16.3 in [FOOO11b]). . Suppose that there exists a ring isomorphism

$$QH_b(M, \Lambda) \cong \Lambda \times Q$$

and  $e \in QH_b(M, \Lambda)$  is the idempotent corresponding to the unit of the first factor of the right hand side. Then the function

$$\mu_e^b : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R}$$

is a homogeneous Calabi quasi-morphism.

From standard properties of spectral invariants (Theorem 7.8 in [FOOO11b]),  $\mu_e^b$  has two additional properties (Theorem 14.1 in [FOOO11b]):

- (1) (Lipschitz continuity) There exists a constant  $C \geq 0$  such that for any  $\tilde{\psi}, \tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$ ,

$$|\mu_e^b(\tilde{\psi}) - \mu_e^b(\tilde{\phi})| \leq C\|\tilde{\psi}\tilde{\phi}^{-1}\|.$$

- (2) (Symplectic invariance) For all  $\psi \in \text{Symp}_0(M, \omega)$ ,

$$\mu_e^b(\tilde{\phi}) = \mu_e^b(\psi \circ \tilde{\phi} \circ \psi^{-1}).$$

Here  $C \leq \text{vol}(M)$  is easily proved as in Proposition 3.5 of [EP03].

On the other hand, *symplectic quasi-states* are also constructed by using (bulk deformed) spectral invariants. Let  $C^0(M)$  be the set of continuous functions on  $M$ .

**Definition 2.2** (Section 3 in [EP06]). A functional  $\zeta : C^0(M) \rightarrow \mathbb{R}$  is called symplectic quasi-state if  $\zeta$  satisfies the following:

- (1) (Normalization)  $\zeta(1) = 1$ .
- (2) (Monotonicity)  $\zeta(F_1) \leq \zeta(F_2)$  for any  $F_1 \leq F_2$ .
- (3) (Homogeneity)  $\zeta(\lambda F) = \lambda \zeta(F)$  for any  $\lambda \in \mathbb{R}$ .
- (4) (Strong quasi-additivity) If smooth functions  $F$  and  $G$  are Poisson commutative:  $\{F, G\} = 0$ , then  $\zeta(F + G) = \zeta(F) + \zeta(G)$ .
- (5) (Vanishing) If  $\text{supp } F$  is displaceable, then  $\zeta(F) = 0$ .
- (6) (Symplectic invariance)  $\zeta(F) = \zeta(F \circ \psi)$  for any  $\psi \in \text{Symp}_0(M, \omega)$ .

By using the bulk deformed spectral invariant  $\rho^b(\cdot; e)$ , a functional  $\zeta_e^b : C^\infty(M) \rightarrow \mathbb{R}$  is defined by

$$\zeta_e^b(H) := - \lim_{m \rightarrow +\infty} \frac{\rho^b(mH; e)}{m}.$$

This functional  $\zeta_e^b$  extends to a functional on  $C^0(M)$  as follows. We recall the relation between  $\zeta_e^b$  and  $\mu_e^b$  (see Section 14 [FOOO11b]). For any  $H \in C^\infty([0, 1] \times M)$ , by the *shift property* of spectral invariant, we have

$$(2.2) \quad \rho^b(\tilde{\phi}_H; e) = \rho^b(H; e) + \frac{1}{\text{vol}(M)} \text{Cal}_M(H),$$

where  $\text{Cal}_M(H)$  is defined by

$$\text{Cal}_M(H) := \int_0^1 dt \int_{M^{2n}} H_t \omega^n.$$

Since  $(\tilde{\phi}_H)^m = \tilde{\phi}_{mH}$  for any autonomous Hamiltonian  $H$ , the following relation is obtained from (2.2)

$$\zeta_e^b(H) = \frac{1}{\text{vol}(M)} \left( -\mu_e^b(\tilde{\phi}_H^1) + \text{Cal}_M(H) \right).$$

By the Lipschitz continuity of  $\mu_e^b$ , we can extend  $\zeta_e^b$  to a functional on  $C^0(M)$ . From the same argument in Section 6 in [EP06], this functional

$\zeta_e^{\mathfrak{b}} : C^0(M) \rightarrow \mathbb{R}$  becomes a symplectic quasi-state if one takes an idempotent  $e$  from a field factor of  $QH_{\mathfrak{b}}(M, \Lambda)$  as in Theorem 2.1.

In this paper, we define *superheavy subsets* as follows.

**Definition 2.3.** Let  $\zeta$  be a symplectic quasi-state on  $(M, \omega)$ . A closed subset  $X \subset M$  is called  $\zeta$ -superheavy if for all  $H \in C^0(M)$

$$\min_X H \leq \zeta(H) \leq \max_X H.$$

It is immediately proved that any  $\zeta$ -superheavy subsets must intersect each other and non-displaceable (see [EP09] for details).

**2.2. Brief review of FOOO's results.** In [FOOO12], Fukaya-Oh-Ohta-Ono computed the full *potential function*, which is a “generating function of open-closed Gromov-Witten invariant”, of some Lagrangian tori in  $S^2 \times S^2$  and they proved superheaviness of these tori in [FOOO11b]. In this section, we briefly describe the construction of their superheavy tori.

Let  $F_2(0)$  be a symplectic toric orbifold whose moment polytope  $P$  is given by

$$P := \{(u_1, u_2) \in \mathbb{R}^2 \mid 0 \leq u_1 \leq 2, 0 \leq u_2 \leq 1 - \frac{1}{2}u_1\}.$$

We denote by  $\pi : F_2(0) \rightarrow P$  the moment map, and denote by  $L(u)$  a Lagrangian torus fiber over an interior point  $u \in \text{Int}(P)$ . Then  $F_2(0)$  has one singular point which corresponds to the point  $(0, 1)$  in  $P$ . They constructed a symplectic manifold  $\hat{F}_2(0)$  which is symplectomorphic to  $(S^2 \times S^2, \frac{1}{2}\omega_{std} \oplus \frac{1}{2}\omega_{std})$ , by replacing a neighborhood of the singularity with a cotangent disk bundle of  $S^2$  (for details, see Section 4 [FOOO12]). Under the smoothing, Lagrangian torus fiber  $L(u)$  is sent to a Lagrangian torus in  $S^2 \times S^2$ . In particular, we denote by  $T_\tau$  ( $0 < \tau \leq \frac{1}{2}$ ) this torus corresponding to  $L((\tau, 1-\tau)) \subset F_2(0)$ .

For these Lagrangian tori  $T_\tau \subset S^2 \times S^2$ , they obtained the following.

**Theorem 2.2** (Fukaya-Oh-Ohta-Ono [FOOO11b]). *For any  $0 < \tau \leq 1/2$ , there exist an element  $\mathfrak{b}(\tau) \in H^{even}(M, \Lambda_0)$  and idempotents  $e_\tau$  and  $e_\tau^0$ , each of which is an idempotent of a field factor of  $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$  such that*

- (1)  $T_\tau$  is  $\mu_{e_\tau}^{\mathfrak{b}(\tau)}$ -superheavy and  $T_{\frac{1}{2}}$  is  $\mu_{e_\tau^0}^{\mathfrak{b}(\tau)}$ -superheavy.
- (2)  $S_{eq}^1 \times S_{eq}^1$  is  $\mu_e^{\mathfrak{b}(\tau)}$ -superheavy for any idempotent  $e$  of a field factor of  $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$ . In particular,

$$\psi(T_\tau) \cap (S_{eq}^1 \times S_{eq}^1) \neq \emptyset$$

for any symplectic diffeomorphism  $\psi$  on  $S^2 \times S^2$ .

Here  $\mu_{e_\tau}^{\mathfrak{b}(\tau)}$  and  $\mu_{e_\tau^0}^{\mathfrak{b}(\tau)}$  denote homogeneous Calabi quasi-morphisms associated to the idempotents  $e_\tau, e_\tau^0 \in QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$  respectively (see Theorem 2.1).

**Remark 2.1.** (1) In [FOOO11b], (1) is Theorem 23.4 (2), and (2) is Theorem 1.13.

- (2) The notion of  $\mu_e^b$ -superheavy is defined in Definition 18.5 of [FOOO11b] and they remark as Remark 18.6 that  $\mu_e^b$ -superheavyness implies  $\zeta_e^b$ -superheavyness. In this paper, we need only to use  $\zeta_e^b$ -superheavyness.
- (3) The quasi-morphisms  $\mu_{e_\tau}^{b(\tau)}$  and  $\mu_{e_\tau^0}^{b(\tau)}$  descend to homogeneous Calabi quasi-morphisms on  $\text{Ham}(S^2 \times S^2)$  as in [EP03].

Hereafter, we use only above homogeneous Calabi quasi-morphisms

$$\mu_{e_\tau}^{b(\tau)} : \text{Ham}(S^2 \times S^2) \rightarrow \mathbb{R}$$

with  $0 < \tau < 1/2$  and denote them by  $\mu^\tau$ .

**2.3. Pullback of the quasi-morphism  $\mu^\tau$ .** To obtain quasi-morphisms on  $\text{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$ , we define a conformally symplectic embedding  $\Theta_\delta : B^2 \times B^2 \hookrightarrow S^2 \times S^2$  for each Lagrangian submanifold  $L_\delta \subset B^2 \times B^2$ .

For each  $1/2 < \delta \leq 1$ , we define a conformally symplectic embedding  $\theta_\delta : (B^2, 2\omega_0) \hookrightarrow (S^2, \frac{1}{2}\omega_{std}) \cong (\mathbb{C}P^1, \omega_{FS})$  by

$$\theta_\delta(z) := [\sqrt{1 - \delta|z|^2} : \sqrt{\delta}z],$$

where we identify the projective space with a unit sphere by using a stereographic projection with respect to  $(1, 0, 0) \in S^2 \subset \mathbb{R}^3$  after regarding the plane  $\{v = (v_1, v_2, v_3) \in \mathbb{R}^3 \mid v_1 = 0\}$  as the complex plane  $\mathbb{C}$ . We note that  $\theta_\delta^*(\frac{1}{2}\omega_{std}) = \delta\omega_0$  and the image of  $\theta_\delta$  is  $\{v \in S^2 \mid v_1 < 2\delta - 1\}$ . Moreover, by the map  $\theta_\delta$ , the circle  $T_\delta \subset B^2$  is mapped onto the equator  $S_0^1 := \{v \in S^2 \mid v_1 = 0\}$  and the real form  $\text{Re}(B^2)$  is mapped into the equator  $S_{eq}^1 := \{v \in \mathbb{R}^3 \mid v_3 = 0\} \subset S^2$ .

Using this conformally symplectic embedding, we define  $\Theta_\delta : B^2 \times B^2 \hookrightarrow S^2 \times S^2$  by

$$(2.3) \quad \Theta_\delta := \theta_\delta \times \theta_\delta : (B^2 \times B^2, \bar{\omega}_0) \hookrightarrow (S^2 \times S^2, \bar{\omega}_{std})$$

where  $\bar{\omega}_{std}$  denotes the symplectic structure  $\frac{1}{2}\omega_{std} \oplus \frac{1}{2}\omega_{std}$  on  $S^2 \times S^2$ . This is a conformally symplectic embedding for each  $1/2 < \delta \leq 1$ . Indeed, it is obvious

$$\Theta_\delta^* \bar{\omega}_{std} = \delta \bar{\omega}_0.$$

For a time-dependent Hamiltonian  $F$  on  $B^2 \times B^2$ , we define a Hamiltonian  $F \circ \Theta_\delta^{-1}$  on  $S^2 \times S^2$  by

$$F \circ \Theta_\delta^{-1}(x) := \begin{cases} F(t, \Theta_\delta^{-1}(x)) & (x \in \text{Im}(\Theta_\delta)) \\ 0 & (x \notin \text{Im}(\Theta_\delta)). \end{cases}$$

Since  $\Theta_\delta$  is a conformally symplectic embedding, we obtain

$$\phi_{\delta F \circ \Theta_\delta^{-1}}^1 = \Theta_\delta \phi_F^1 \Theta_\delta^{-1}.$$

Thus,  $\Theta_\delta \phi \Theta_\delta^{-1}$  is a Hamiltonian diffeomorphism on  $S^2 \times S^2$  for any  $\phi \in \text{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$ .

We define a family of quasi-morphisms  $\mu_\delta^\tau : \text{Ham}_c(B^2 \times B^2, \bar{\omega}_0) \rightarrow \mathbb{R}$  by

$$(2.4) \quad \mu_\delta^\tau(\phi) := \frac{\delta^{-1}}{\text{vol}(S^2 \times S^2)} (-\mu^\tau(\Theta_\delta \phi \Theta_\delta^{-1}) + \text{Cal}_{\Theta_\delta(B^2 \times B^2)}(\Theta_\delta \phi \Theta_\delta^{-1})),$$

where  $\mu^\tau$  are Fukaya-Oh-Ohta-Ono's quasi-morphisms in Section 2.2 and  $\text{Cal}_{\Theta_\delta(B^2 \times B^2)}$  is the Calabi morphism on  $\text{Ham}_{\Theta_\delta(B^2 \times B^2)}(S^2 \times S^2, \bar{\omega}_{std})$  in Section 2.1. The symplectic structure  $\bar{\omega}_{std}$  is exact on  $\Theta_\delta(B^2 \times B^2)$ , hence the right hand side of (2.4) does not depend on the choice of the Hamiltonian generating  $\phi$ . Moreover, by the definition, it turns out that  $\mu_\delta^\tau$  are quasi-morphisms.

To obtain another expression of  $\mu_\delta^\tau$ , we define  $\zeta^\tau : C^\infty([0, 1] \times S^2 \times S^2) \rightarrow \mathbb{R}$  as the following :

$$\zeta^\tau(H) := - \lim_{n \rightarrow \infty} \frac{\rho^{b(\tau)}(H^{\#n}; e_\tau)}{n},$$

where we denote by  $H_1 \# H_2$  the concatenation of two Hamiltonian  $H_1$  and  $H_2$  :

$$H_1 \# H_2(t, x) := \begin{cases} \chi'(t) H_1(\chi(t), x) & 0 \leq t \leq 1/2 \\ \chi'(t - 1/2) H_2(\chi(t), x) & 1/2 \leq t \leq 1 \end{cases}$$

for a smooth function  $\chi : [0, 1/2] \rightarrow [0, 1]$  with  $\chi' \geq 0$  and  $\chi \equiv 0$  near  $t = 0$ ,  $\chi \equiv 1$  near  $t = 1/2$ . Note that this definition is independent of the function  $\chi$  since the spectral invariant  $\rho^{b(\tau)}$  has homotopy invariance property.

By the definition and (2.2), one can check that

$$(2.5) \quad \zeta^\tau(H) = \frac{1}{\text{vol}(S^2 \times S^2)} (-\mu^\tau(\phi_H^1) + \text{Cal}_{S^2 \times S^2}(H))$$

for any time-dependent Hamiltonian  $H$  and the restriction of  $\zeta^\tau$  to autonomous Hamiltonians corresponds to the bulk-deformed quasi-state  $\zeta_{e_\tau}^{b(\tau)}$  which is associated to  $\mu^\tau = \mu_{e_\tau}^{b(\tau)}$ .

Therefore, by (2.4) and (2.5), we obtain the following expression of  $\mu_\delta^\tau$ .

**Lemma 2.1.**

$$\mu_\delta^\tau(\phi_F^1) = \delta^{-1} \zeta^\tau(\delta F \circ \Theta_\delta^{-1}).$$

### 3. PROPERTIES OF QUASI-MORPHISMS $\mu_\delta^\tau$ ON $\text{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$

In this section, we prove some properties of the quasi-morphisms  $\mu_\delta^\tau$  by following procedures in [Se14]. Since Proposition 3.1 and Proposition 3.2 are proved by using only standard properties of Calabi quasi-morphisms, two proofs are the same as in [Se14]. However the proof of Proposition 3.3 depends on some properties of Lagrangian submanifolds and ambient spaces, thus we need to modify the proof slightly for our Lagrangian submanifolds  $L_\delta \subset B^2 \times B^2$ .

**Proposition 3.1.** *For any  $0 < \tau < 1/2$  and  $1/2 < \delta \leq 1$ , we have*

- (1)  $|\mu_\delta^\tau(\phi)| \leq C_\delta \|\phi\|$ , where  $C_\delta$  is a positive constant.

(2) If a time-dependent Hamiltonian  $H_t$  on  $B^2 \times B^2$  is supported in a displaceable subset for any time  $t \in [0, 1]$  then we have

$$\mu_\delta^\tau(\phi_H^1) = 0.$$

*Proof.* Let  $\phi_F^1$  be an element in  $\text{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$ . Since the quasi-morphisms  $\mu^\tau$  have Lipschitz continuity property with respect to the Hofer norm on  $\text{Ham}(S^2 \times S^2, \bar{\omega}_{std})$  and  $\Theta_\delta \phi_F^1 \Theta_\delta^{-1} = \phi_{\delta F \circ \Theta_\delta}^1$ , we obtain

$$|\mu^\tau(\Theta_\delta \phi_F^1 \Theta_\delta^{-1})| \leq \text{vol}(S^2 \times S^2) \|\phi_{\delta F \circ \Theta_\delta}^1\|.$$

By the definition of the Hofer norm, it turns out that

$$\|\phi_{\delta F \circ \Theta_\delta}^1\| \leq \delta \|\phi_F^1\|.$$

Hence, we have

$$|\mu^\tau(\Theta_\delta \phi_F^1 \Theta_\delta^{-1})| \leq \delta \text{vol}(S^2 \times S^2) \|\phi_F^1\|.$$

On the other hand, an easily calculation shows that

$$\text{Cal}_{\Theta_\delta(B^2 \times B^2)}(\Theta_\delta \phi_F^1 \Theta_\delta^{-1}) = \delta^3 \int_0^1 dt \int_{B^2 \times B^2} F(t, x) \bar{\omega}_0^2.$$

As a result, we can obtain the following:

$$|\text{Cal}_{\Theta_\delta(B^2 \times B^2)}(\Theta_\delta \phi_F^1 \Theta_\delta^{-1})| \leq \delta^3 \text{vol}(B^2 \times B^2) \|\phi_F^1\|.$$

Consequently, it turns out that

$$\begin{aligned} |\mu_\delta^\tau(\phi)| &\leq \frac{\delta^{-1}}{\text{vol}(S^2 \times S^2)} \left( |\mu^\tau(\Theta_\delta \phi \Theta_\delta^{-1})| + |\text{Cal}_{\Theta_\delta(B^2 \times B^2)}(\Theta_\delta \phi \Theta_\delta^{-1})| \right) \\ &\leq (1 + \delta^2) \|\phi\|. \end{aligned}$$

Thus (1) is proved.

The property (2) follows immediately from Calabi-property of  $\mu^\tau$ . Indeed, two terms in the definition of  $\mu_\delta^\tau$  are canceled each other.  $\square$

Let  $X \subset S^2 \times S^2$  be a  $\zeta_{e_\tau}^{b(\tau)}$ -superheavy subset. By definition, we have

$$\min_X H \leq \zeta_{e_\tau}^{b(\tau)}(H) \leq \max_X H$$

for all autonomous Hamiltonians  $H$  on  $S^2 \times S^2$ . One can obtain the same inequality for  $\zeta^\tau : C^\infty([0, 1] \times S^2 \times S^2) \rightarrow \mathbb{R}$  if a closed subset  $X \subset S^2 \times S^2$  is  $\zeta_{e_\tau}^{b(\tau)}$ -superheavy. More precisely, for all time-dependent Hamiltonians  $H$  on  $S^2 \times S^2$ , we have

$$(3.1) \quad \min_{[0,1] \times X} H \leq \zeta^\tau(H) \leq \max_{[0,1] \times X} H.$$

This is easily proved as mentioned in [Se14] without the detail. Indeed, we can take two autonomous Hamiltonians  $H_{\min}, H_{\max}$  for any time-dependent Hamiltonian  $H$  such that  $H_{\min} \equiv \min_{[0,1] \times X} H$ ,  $H_{\max} \equiv \max_{[0,1] \times X} H$  on

$X$  and  $H_{\min} \leq H \leq H_{\max}$  on  $S^2 \times S^2$ . By applying the anti<sup>1</sup>-monotonicity property of  $\rho^{b(\tau)}$  (i.e.  $H \leq K \Rightarrow \rho^{b(\tau)}(H; e_\tau) \geq \rho^{b(\tau)}(K; e_\tau)$ , see Theorem 9.1 in [FOOO11b]) and the fact  $H \leq K$  implies  $H^{\#n} \leq K^{\#n}$  to above Hamiltonians  $H_{\min}, H, H_{\max}$ , we can obtain (3.1) immediately.

From Lemma 2.1 and this inequality (3.1), we obtain the following.

**Proposition 3.2.** *Suppose a closed subset  $X \subset S^2 \times S^2$  is  $\zeta_{e_\tau}^{b(\tau)}$ -superheavy and  $F$  is any compactly supported time-dependent Hamiltonian on the bi-disks  $B^2 \times B^2$  such that  $F \circ \Theta_\delta^{-1}|_X \equiv c$ , then*

$$\mu_\delta^\tau(\phi_F^1) = c.$$

Proposition 3.3 is the most important to obtain unboundedness of  $(\mathcal{L}(L_\delta), d)$ . In [Kh09], Khanevsky proved the similar property and obtained the unboundedness for the case where the ambient space is two-dimensional open ball. In [Se14], by a different proof, Seyfaddini also obtained the similar property for  $(\mathcal{L}(Re(B^{2n})), d)$ .

**Proposition 3.3.** *If two Hamiltonian diffeomorphisms  $\phi, \psi \in \text{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$  satisfy*

$$\phi(L_\delta) = \psi(L_\delta),$$

*then we have*

$$|\mu_\delta^\tau(\phi) - \mu_\delta^\tau(\psi)| \leq \frac{D_{\mu^\tau}}{\delta \text{vol}(S^2 \times S^2)} \text{ for all } \frac{1}{2} < \delta \leq 1, 0 < \tau < \frac{1}{2},$$

*where  $D_{\mu^\tau}$  denotes the defect of  $\mu^\tau$ .*

We prove this proposition by slightly modifying Seyfaddini's proof.

*Proof.* Throughout the proof, we fix  $\delta, \tau$  with  $1/2 < \delta \leq 1, 0 < \tau < 1/2$ , respectively. From the definition of  $\mu_\delta^\tau$  and its homogeneity we obtain that

$$\begin{aligned} & |\mu_\delta^\tau(\phi^{-1}\psi) + \mu_\delta^\tau(\phi) - \mu_\delta^\tau(\psi)| \\ &= |\mu_\delta^\tau(\phi^{-1}\psi) - \mu_\delta^\tau(\phi^{-1}) - \mu_\delta^\tau(\psi)| \\ &= \frac{1}{\delta \text{vol}(S^2 \times S^2)} |\mu^\tau(\Theta_\delta \phi^{-1}\psi \Theta_\delta^{-1}) - \mu^\tau(\Theta_\delta \phi^{-1}\Theta_\delta^{-1}) - \mu^\tau(\Theta_\delta \psi \Theta_\delta^{-1})| \\ &\leq \frac{D_{\mu^\tau}}{\delta \text{vol}(S^2 \times S^2)}. \end{aligned}$$

Consequently, it is sufficient to prove the proposition that  $\mu_\delta^\tau(\phi)$  vanishes for Hamiltonian diffeomorphisms  $\phi$  satisfying  $\phi(L_\delta) = L_\delta$ .

Now we take any Hamiltonian  $F \in C_c^\infty([0, 1] \times (B^2 \times B^2))$  and assume the Hamiltonian diffeomorphism  $\phi_F^1$  preserves the Lagrangian submanifold  $L_\delta$ .

For  $0 < s \leq 1$ , we define a diffeomorphism  $a_s : B^2 \times B^2(s) \rightarrow B^2 \times B^2$  by

$$a_s(z_1, z_2) := (z_1, \frac{z_2}{s}).$$

---

<sup>1</sup>Fukaya-Oh-Ohta-Ono used different sign conventions from [EP03, EP06, EP09] (see Remark 4.17 in [FOOO11b]).

Using this map, we define a compactly supported symplectic diffeomorphism  $\psi_s$  for each  $0 < s \leq 1$ :

$$\psi_s := \begin{cases} a_s^{-1} \phi_F^1 a_s & |z_2| \leq s \\ id & |z_2| \geq s \end{cases}.$$

As compactly supported cohomology group  $H_c^1(B^2 \times B^2; \mathbb{R}) = 0$  and  $\bar{\omega}_0$  is exact on  $B^2 \times B^2$ , any isotopy of compactly supported Symplectic diffeomorphisms on  $(B^2 \times B^2, \bar{\omega}_0)$  is a compactly supported Hamiltonian isotopy. Thus, for each  $0 < s \leq 1$ , we can take a time-dependent Hamiltonian  $F^s \in C_c^\infty([0, 1] \times B^2 \times B^2)$  such that  $\psi_s = \phi_{F^s}^1$ .

This Hamiltonian diffeomorphisms  $\psi_s$  have the following properties:

- (1)  $\psi_1 = \phi_{F^1}^1 = \phi_F^1$ ,
- (2)  $\psi_s$  preserves  $L_\delta$  for each  $0 < s \leq 1$ ,
- (3) There exists a compact subset  $K_s$  in  $B^2$  such that  $F^s$  is supported in  $K_s \times B^2(s) \subset B^2 \times B^2$  for each  $0 < s \leq 1$ .

Hereafter we fix sufficiently small  $\epsilon > 0$  such that  $K_\epsilon \times B^2(\epsilon)$  is displaceable inside the bi-disks  $B^2 \times B^2$ . By Proposition 3.1 (2), it follows that

$$(3.2) \quad \mu_\delta^\tau(\psi_\epsilon) = 0.$$

We take a time-dependent Hamiltonian  $H \in C_c^\infty([0, 1] \times B^2 \times B^2)$  so that  $\phi_H^t := \psi_\epsilon^{-1} \psi_{t(1-\epsilon)+\epsilon}$  for  $0 \leq t \leq 1$ . In particular, we have the time-one map  $\phi_H^1 = \psi_\epsilon^{-1} \phi_F^1$  by the above property (1).

We note that Hamiltonian vector field  $X_{H_t}$  is tangent to the Lagrangian submanifold  $L_\delta$  since  $\phi_H^t$  preserves  $L_\delta$ . Consequently, for each  $t \in [0, 1]$ ,  $H_t = H(t, \cdot)$  is constant on  $L_\delta$ . Because of this and non-compactness of  $L_\delta$ , the restriction of  $H_t$  to  $L_\delta$  is 0 for all  $t \in [0, 1]$ . Since  $L_\delta = T_\delta \times Re(B^2)$  is mapped into  $S_0^1 \times S_{eq}^1$  by  $\Theta_\delta$ , hence  $H \circ \Theta_\delta^{-1}$  vanishes on a torus  $S_0^1 \times S_{eq}^1$ . On the other hand  $S_0^1 \times S_{eq}^1$  is  $\zeta_{e_\tau}^{b(\tau)}$ -superheavy by Fukaya-Oh-Ohta-Ono's result (Theorem 2.2), therefore we have

$$(3.3) \quad \mu_\delta^\tau(\phi_H^1) = 0.$$

Here we used Proposition 3.2.

As a consequence of these two equalities (3.2), (3.3) and quasi-additivity of  $\mu_\delta^\tau$ , it follows that

$$|\mu_\delta^\tau(\phi_F^1)| = |\mu_\delta^\tau(\phi_F^1) - \mu_\delta^\tau(\psi_\epsilon) - \mu_\delta^\tau(\phi_H^1)| \leq \frac{D_{\mu^\tau}}{\delta \text{vol}(S^2 \times S^2)}.$$

Because  $(\phi_F^1)^n$  preserves  $L_\delta$  for any  $n \in \mathbb{N}$ , we can apply the same argument to  $(\phi_F^1)^n$  and obtain  $|\mu_\delta^\tau((\phi_F^1)^n)| \leq \delta^{-1} \text{vol}(S^2 \times S^2)^{-1} D_{\mu^\tau}$ . Since  $\mu_\delta^\tau$  is a homogeneous quasi-morphism, we have

$$\mu_\delta^\tau(\phi_F^1) = 0.$$

□

By applying Proposition 3.1 (1) and Proposition 3.3, we obtain the following.

**Proposition 3.4.** *For any  $\phi \in \text{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$  and any  $\frac{1}{2} < \delta \leq 1$ ,  $0 < \tau < \frac{1}{2}$ , the following inequality holds.*

$$\frac{\mu_\delta^\tau(\phi) - \delta^{-1} \text{vol}(S^2 \times S^2)^{-1} D_{\mu^\tau}}{C_\delta} \leq d(L_\delta, \phi(L_\delta)),$$

where  $D_{\mu^\tau}$  is as above.

*Proof.* We take any  $\psi \in \text{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$  satisfying  $\phi(L_\delta) = \psi(L_\delta)$ . From Proposition 3.3, we obtain the following inequality.

$$|\mu_\delta^\tau(\phi) - \mu_\delta^\tau(\psi)| \leq \frac{D_{\mu^\tau}}{\delta \text{vol}(S^2 \times S^2)}.$$

By using Proposition 3.1 (1), we have

$$|\mu_\delta^\tau(\phi)| - \frac{D_{\mu^\tau}}{\delta \text{vol}(S^2 \times S^2)} \leq |\mu_\delta^\tau(\psi)| \leq C_\delta \|\psi\|.$$

Therefore, by definition of the metric  $d$ , we obtain the following inequality:

$$|\mu_\delta^\tau(\phi)| - \frac{D_{\mu^\tau}}{\delta \text{vol}(S^2 \times S^2)} \leq C_\delta \cdot d(L_\delta, \psi(L_\delta)).$$

□

#### 4. CONSTRUCTION OF $\Phi_\delta : C_c^\infty((0, 1)) \rightarrow \mathcal{L}(L_\delta)$

**4.1. Locations of FOOO's superheavy tori.** To construct a mapping  $\Phi_\delta : C_c^\infty((0, 1)) \rightarrow \mathcal{L}(L_\delta)$  in Theorem 1.2, we describe the locations of Fukaya-Oh-Ohta-Ono's Lagrangian superheavy tori by following Oakley-Usher's result. Let us recall their description. In [OU13], they constructed a symplectic toric orbifold  $\mathcal{O}$  which is isomorphic to  $F_2(0)$  as symplectic toric orbifolds by gluing  $S^2 \times S^2 \setminus \bar{\Delta}$  to  $B^4/\{\pm 1\}$ . Here  $\bar{\Delta}$  denotes anti-diagonal of  $S^2 \times S^2$  and  $B^4$  is a four dimensional open ball. The moment map  $\pi : \mathcal{O} \rightarrow \mathbb{R}^2$ , which has the same moment polytope  $P$  of  $F_2(0)$  in Section 2.2, is expressed on  $S^2 \times S^2 \setminus \bar{\Delta}$  by

$$\pi(v, w) = \left( \frac{1}{2}|v + w| + \frac{1}{2}(v + w) \cdot e_1, 1 - \frac{1}{2}|v + w| \right) \in \mathbb{R}^2$$

for  $(v, w) \in S^2 \times S^2 \setminus \bar{\Delta}$  and  $e_1 := (1, 0, 0)$ . Therefore one can consider a torus fiber  $L(u) \subset F_2(0)$  as  $\pi^{-1}(u) \subset S^2 \times S^2 \setminus \bar{\Delta}$  for any interior point  $u$  in the moment polytope.

By replacing  $B^4/\{\pm 1\}$  by the unit disk cotangent bundle  $D_1^* S^2$ , they obtained a smoothing  $\Pi : \hat{\mathcal{O}} \rightarrow \mathcal{O}$  which maps the zero-section of  $D_1^* S^2$  to the singularity of  $\mathcal{O}$  and whose restriction to  $S^2 \times S^2 \setminus \bar{\Delta}$  is the identity mapping. Moreover they gave an explicit symplectic morphism  $\hat{\mathcal{O}} \xrightarrow{\sim} S^2 \times S^2$  which is the identity mapping on  $S^2 \times S^2 \setminus \bar{\Delta}$ . Hence above tori  $\pi^{-1}(u)$  are invariant under the smoothing and the symplectic morphism  $\hat{\mathcal{O}} \xrightarrow{\sim} S^2 \times S^2$ .

Using this construction, Oakley-Usher proved that the Entov-Polterovich's exotic monotone torus in [EP09] is Hamiltonian isotopic to the Fukaya-Oh-Ohta-Ono's torus over  $(1/2, 1/2)$  (for details, see the proof of Proposition 2.1 [OU13]).

**Proposition 4.1** (Oakley-Usher [OU13]). *Fukaya-Oh-Ohta-Ono's superheavy Lagrangian tori  $T_\tau$  can be expressed as*

$$T_\tau = \left\{ (v, w) \in S^2 \times S^2 \mid \frac{1}{2}|v + w| + \frac{1}{2}(v + w) \cdot e_1 = \tau, \ 1 - \frac{1}{2}|v + w| = 1 - \tau \right\},$$

where the parameter  $\tau$  is in  $(0, 1/2]$ . In particular, the Lagrangian torus  $T_{1/2}$  is Entov-Polterovich's exotic monotone torus.

The following corollary is proved by an easily calculation.

**Corollary 4.1.** *The image of  $i$ -th projection  $\text{pr}_i : S^2 \times S^2 \rightarrow S^2$  ( $i = 1, 2$ ) is*

$$(4.1) \quad \text{pr}_i(T_\tau) = \left\{ v \in S^2 \mid |v \cdot e_i| \leq \sqrt{1 - \tau^2} \right\},$$

where  $\tau$  is  $0 < \tau \leq 1/2$ .

By this corollary and the definition of the conformally symplectic embedding  $\Theta_\delta : B^2 \times B^2 \hookrightarrow S^2 \times S^2$ . We have the following.

**Corollary 4.2.** *For any  $(2 + \sqrt{3})/4 < \delta \leq 1$  there exists a sufficiently small  $\varepsilon_\delta > 0$  such that*

$$\bigcup_{\tau \in I_\delta} T_\tau \subset \Theta_\delta(B^2 \times B^2), \quad I_\delta := [1/2 - \varepsilon_\delta, 1/2].$$

**Remark 4.1.** The condition  $(2 + \sqrt{3})/4 < \delta \leq 1$  in Theorem 1.2 guarantees that the image of  $\Theta_\delta$  contains a continuous subfamily of superheavy tori  $T_\tau \subset S^2 \times S^2$  as in Corollary 4.2. However, for any  $1/2 < \delta \leq 1$ , it is likely that there exist  $\phi_\delta \in \text{Ham}(S^2 \times S^2)$  such that the image of  $\Theta_\delta$  contains  $\cup_{\tau \in I'_\delta} \phi_\delta(T_\tau)$  for some open interval  $I'_\delta \subset (0, 1/2]$ . In this case, we can show Theorem 1.2 under the weaker assumption  $1/2 < \delta \leq 1$ .

**4.2. Construction of  $\Phi_\delta$ .** We fix  $\delta$  with  $(2 + \sqrt{3})/4 < \delta \leq 1$  and consider the interval  $I_\delta = [1/2 - \varepsilon_\delta, 1/2]$  in Corollary 4.2. We take a segment  $J_\delta$  in the moment polytope  $P = \pi(\mathcal{O}) \subset \mathbb{R}^2$  defined by

$$J_\delta := \{(\tau, 1 - \tau) \mid \tau \in \text{Int}(I_\delta)\} \subset \text{Int}(P).$$

We denote by  $B^2(u_0; \sqrt{2}\varepsilon_\delta)$  the open disk of which center is  $u_0 := (1/2, 1/2) \in \text{Int}(P)$  and radius is  $\sqrt{2}\varepsilon_\delta$ . We may take and fix a sufficiently small  $\varepsilon_\delta > 0$  so that the open disk  $B^2(u_0; \sqrt{2}\varepsilon_\delta)$  is contained in  $P$  and moreover the inverse image of  $B^2(u_0; \sqrt{2}\varepsilon_\delta)$  under  $\tilde{\pi} := \pi \circ \Pi : \hat{O} \rightarrow P$  is contained in the image of  $\Theta_\delta : B^2 \times B^2 \rightarrow S^2 \times S^2$ .

We identify  $J_\delta$  with an open interval  $(0, 1)$  and will define a map  $\Phi_\delta$  on  $C_c^\infty(J_\delta)$ . First, we extend a function  $f \in C_c^\infty(J_\delta)$  to the function  $f_{B^2}$  on the

open disk  $B^2(u_0; \sqrt{2}\varepsilon_\delta)$  which is constant along the circle centered at  $u_0$ . More explicitly, we define  $f_{B^2} : B^2(u_0; \sqrt{2}\varepsilon_\delta) \rightarrow \mathbb{R}$  by

$$f_{B^2}(u) := f((|u - u_0|/\sqrt{2}, 1 - |u - u_0|/\sqrt{2})), \quad u \in B^2(u_0; \sqrt{2}\varepsilon_\delta) \subset \text{Int}(P).$$

We define  $\tilde{f} \in C_c^\infty(B^2 \times B^2)$  for  $f \in C_c^\infty(J_\delta)$  as the pull-back:

$$(4.2) \quad \tilde{f} := \Theta_\delta^* \tilde{\pi}^* f_{B^2}.$$

By the construction, the restriction of  $\tilde{f}$  on  $\Theta_\delta^{-1}(T_\tau)$  is constantly equal to  $f(\tau)$  for all  $1/2 - \varepsilon_\delta < \tau < 1/2$ .

**Definition 4.1.** For any  $(2 + \sqrt{3})/4 < \delta \leq 1$ , we define  $\Phi_\delta : C_c^\infty((0, 1)) \rightarrow \mathcal{L}(L_\delta)$  by the following expression:

$$\Phi_\delta(f) := \phi_{\tilde{f}}^1(L_\delta),$$

where we regard  $f$  as an element in  $C_c^\infty(J_\delta)$ .

For the proof of Theorem 1.2, we prove the next lemma.

**Lemma 4.1.** *For any  $f, g \in C_c^\infty((1/2 - \varepsilon_\delta, 1/2))$  there exists a constant  $1/2 - \varepsilon_\delta < \tau' < 1/2$  such that*

$$|\mu_\delta^{\tau'}(\phi_{\tilde{f}-\tilde{g}}^1)| = \|f - g\|_\infty,$$

where  $\delta$  is  $(2 + \sqrt{3})/4 < \delta \leq 1$ .

*Proof.* For any  $f, g \in C_c^\infty((1/2 - \varepsilon_\delta, 1/2))$ , there exists  $\tau' \in (1/2 - \varepsilon_\delta, 1/2)$  such that

$$\|f - g\|_\infty = \max |f(x) - g(x)| = |f(\tau') - g(\tau')|.$$

Thus  $\mu_\delta^{\tau'}(\phi_{\tilde{f}-\tilde{g}}^1)$  is equal to  $\|f - g\|_\infty$  because of (4.2) and Proposition 3.2.  $\square$

## 5. PROOF OF THEOREM 1.1 AND THEOREM 1.2.

*proof of Theorem 1.1.* For all  $1/2 < \delta \leq 1$ , the image of  $\Theta_\delta$  contains the torus  $S_0^1 \times S_0^1 \subset (S^2 \times S^2, \bar{\omega}_{std})$ . If we take a Hamiltonian  $H \in C_c^\infty(B^2 \times B^2)$  for any  $h \in \mathbb{R}$  such that  $H \equiv h$  on the torus  $\Theta_\delta^{-1}(S_0^1 \times S_0^1)$ , then we have from Proposition 3.2 and  $\zeta_{e^\tau}^{b(\tau)}$ -superheavyness of  $S_0^1 \times S_0^1$

$$\mu_\delta^\tau(\phi_H^1) = h,$$

where we fix any  $\tau \in (0, \frac{1}{2})$ . By applying Proposition 3.4, we obtain

$$\frac{h - \delta^{-1} \text{vol}(S^2 \times S^2)^{-1} D_{\mu^\tau}}{C_\delta} \leq d(L_\delta, \phi(L_\delta)).$$

Since  $h$  is an arbitrary constant, Theorem 1.1 is proved.  $\square$

Theorem 1.1 is proved by using a single quasi-morphism  $\mu_\delta^\tau$  on  $\text{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$ .

On the other hand, to prove Theorem 1.2, it is necessary that the image  $\Theta_\delta(B^2 \times B^2)$  contains a continuous subfamily of superheavy tori  $\phi_\delta(T_\tau) \subset S^2 \times S^2$  for some  $\phi_\delta \in \text{Ham}(S^2 \times S^2)$  as mentioned in Remark 4.1.

In this paper, we consider the case  $\phi_\delta = id$ . Then we need to use the parameter  $\delta$  of our Lagrangian submanifolds  $L_\delta$  with  $(2 + \sqrt{3})/4 < \delta \leq 1$  as in Corollary 4.2.

*proof of Theorem 1.2.* First, we will prove the left-hand side inequality. For any  $f, g \in C_c^\infty((1/2 - \varepsilon_\delta, 1/2))$ , we have  $\tilde{f}, \tilde{g} \in C_c^\infty(B^2 \times B^2)$  defined by (4.2). Then we apply Proposition 3.4 to  $\phi_{\tilde{g}}^{-1} \circ \phi_{\tilde{f}}^1 \in \text{Ham}_c(B^2 \times B^2, \bar{\omega}_0)$  to obtain

$$(5.1) \quad \frac{|\mu_\delta^\tau(\phi_{\tilde{g}}^{-1} \circ \phi_{\tilde{f}}^1)| - \delta^{-1} \text{vol}(S^2 \times S^2)^{-1} D_{\mu^\tau}}{C_\delta} \leq d(L_\delta, \phi_{\tilde{g}}^{-1} \circ \phi_{\tilde{f}}^1(L_\delta)),$$

where  $\phi_{\tilde{g}}^{-1}$  is the inverse of  $\phi_{\tilde{g}}^1$ . By the construction of autonomous Hamiltonians  $\tilde{f}, \tilde{g}$  in (4.2), we find that the Poisson bracket  $\{\tilde{f}, \tilde{g}\}_{\bar{\omega}_0}$  vanishes. Thus we have

$$\phi_{\tilde{g}}^{-1} \circ \phi_{\tilde{f}}^1 = \phi_{\tilde{f}-\tilde{g}}^1.$$

Therefore the inequality (5.1) becomes

$$\frac{|\mu_\delta^\tau(\phi_{\tilde{f}-\tilde{g}}^1)| - \delta^{-1} \text{vol}(S^2 \times S^2)^{-1} D_{\mu^\tau}}{C_\delta} \leq d(\phi_{\tilde{g}}^1(L_\delta), \phi_{\tilde{f}}^1(L_\delta)).$$

By Lemma 4.1, we obtain the following inequality:

$$\frac{\|f - g\|_\infty - \delta^{-1} \text{vol}(S^2 \times S^2)^{-1} D_{\mu^{\tau'}}}{C_\delta} \leq d(\Phi_\delta(f), \Phi_\delta(g)),$$

where the constant  $\tau'$  depends on  $f$  and  $g$ . We prove the following lemma in Section 6.

**Lemma 5.1.** *For any bulk-deformation parameter  $\tau \in (0, 1/2)$ , the defect  $D_{\mu^\tau}$  of quasi-morphisms  $\mu^\tau$  satisfies*

$$D_{\mu^\tau} \leq 12.$$

Therefore, we obtain the left-hand side inequality by putting  $D_\delta := \delta^{-1} \text{vol}(S^2 \times S^2)^{-1} \cdot \sup_\tau D_{\mu^\tau}$ .

The right-hand side inequality is proved immediately. Indeed, we can estimate as the following:

$$\begin{aligned} d(\Phi_\delta(f), \Phi_\delta(g)) &= d(L_\delta, \phi_{\tilde{g}}^{-1} \phi_{\tilde{f}}^1(L_\delta)) \leq \|\tilde{f} - \tilde{g}\| \\ &= \|f - g\|. \end{aligned}$$

This completes the proof of Theorem 1.2.  $\square$

## 6. FINITENESS OF $D_{\mu^\tau}$

The estimate in Lemma 5.1 can be obtained by almost the same calculation of Proposition 21.7 in [FOOO11b]. For this reason, we only sketch the outline of the calculation and use the same notation used in [FOOO11b].

*proof of Lemma 5.1.* From Remark 16.8 in [FOOO11b], upper bounds of defects  $D_{\mu^\tau}$  can be taken to be  $-12\mathfrak{v}_T(e_\tau)$ , where  $\mathfrak{v}_T$  is a valuation of bulk-deformed quantum cohomology  $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$ . The proof of Theorem 2.2 (Theorem 23.4 [FOOO11b]) implies that the idempotent  $e_\tau \in QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$  can be taken from one of four idempotents in  $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$  which decompose quantum cohomology as follows:

$$QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda) = \bigoplus_{(\epsilon_1, \epsilon_2) = (\pm 1, \pm 1)} \Lambda \cdot e_{\epsilon_1, \epsilon_2}^\tau.$$

Here the quantum product in  $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2)$  respects this splitting (i.e. it is semi-simple).

Hence, to prove Lemma 5.1, we only have to estimate the maximum valuation of  $e_{\epsilon_1, \epsilon_2}^\tau$ . For this purpose, we regard  $S^2 \times S^2$  as the symplectic toric manifold with the moment polytope:

$$P = \{u = (u_1, u_2) \in \mathbb{R}^2 \mid l_i(u) \geq 0, i = 1, \dots, 4\},$$

where

$$l_1 = u_1, \quad l_2 = u_2, \quad l_3 = -u_1 + 1, \quad l_4 = -u_2 + 1.$$

We denote by  $\partial_i P := \{l_i(u) = 0\}$  each facets of  $P$  and put  $D_i := \pi^{-1}(\partial_i P)$ , where  $\pi : S^2 \times S^2 \rightarrow P \subset \mathbb{R}^2$  is the moment map. In the following, we fix

$$e_0 := PD[S^2 \times S^2], \quad e_1 := PD[D_1], \quad e_2 := PD[D_2], \quad e_3 := PD[D_1 \cap D_2]$$

as basis of  $H^*(S^2 \times S^2; \mathbb{C})$  and denote by  $L(u_0)$  the Lagrangian torus fiber over  $(1/2, 1/2) \in P$ .

The element  $\mathfrak{b}(\tau)$  in Theorem 2.2 is defined by

$$(6.1) \quad \mathfrak{b}(\tau) := aPD[D_1] + aPD[D_2], \quad a := T^{\frac{1}{2}-\tau}.$$

In our case, since  $S^2 \times S^2$  is Fano, the potential function  $\mathfrak{PO}_{\mathfrak{b}(\tau)}$  is determined in terms of the moment polytope data. Hence we obtain the following expression as in the proof of Theorem 23.4 [FOOO11b]

$$\mathfrak{PO}_{\mathfrak{b}(\tau)} = e^a y_1 + e^{-a} y_2 + y_1^{-1} T + y_2^{-1} T,$$

where  $y_1, \dots, y_4$  are formal variables and  $e^a := \sum_{n=0}^{\infty} a^n / n! \in \Lambda_0$  (see Section 3 in [FOOO11a] and Section 20.4 in [FOOO11b] for the definition of potential functions for toric fibers).

By Proposition 1.2.16 in [FOOO10], the *Jacobian ring*  $\text{Jac}(\mathfrak{PO}_{\mathfrak{b}(\tau)}; \Lambda)$  of the potential function  $\mathfrak{PO}_{\mathfrak{b}(\tau)}$ , which is defined as a certain quotient

ring of the Laurent polynomial  $\Lambda[y_1, \dots, y_4, y_1^{-1}, \dots, y_4^{-1}]$  for our case, is decomposed as follows:

$$\text{Jac}(\mathfrak{PO}_{\mathfrak{b}(\tau)}; \Lambda) = \bigoplus_{(\epsilon_1, \epsilon_2) = (\pm 1, \pm 1)} \Lambda \cdot 1_{\epsilon_1, \epsilon_2}^{\tau},$$

where  $1_{\epsilon_1, \epsilon_2}^{\tau}$  is the unit on each component. More explicitly, we have

$$1_{\epsilon_1, \epsilon_2}^{\tau} = \frac{1}{4}[1 + \epsilon_1 e^{\frac{a}{2}} y_1 T^{-\frac{1}{2}} + \epsilon_2 e^{-\frac{a}{2}} y_2 T^{-1/2} + \epsilon_1 \epsilon_2 y_1 y_2 T^{-1}].$$

We denote by  $e_{\epsilon_1, \epsilon_2}^{\tau}$  the idempotent of  $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$  which corresponds to  $1_{\epsilon_1, \epsilon_2}^{\tau}$  under the *Kodaira-Spencer map*:

$$\mathfrak{ks}_{\mathfrak{b}(\tau)} : QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda) \rightarrow \text{Jac}(\mathfrak{PO}_{\mathfrak{b}(\tau)}; \Lambda),$$

which is a ring isomorphism (see Theorem 20.18 in [FOOO11b]). The same calculation as in Remark 1.3.1 [FOOO10] shows that the Kodaira-Spencer map  $\mathfrak{ks}_{\mathfrak{b}(\tau)}$  maps the basis of  $QH_{\mathfrak{b}(\tau)}(S^2 \times S^2; \Lambda)$  to the following:

$$\mathfrak{ks}_{\mathfrak{b}(\tau)}(e_0) = [1], \mathfrak{ks}_{\mathfrak{b}(\tau)}(e_1) = [e^a y_1], \mathfrak{ks}_{\mathfrak{b}(\tau)}(e_2) = [e^{-a} y_2], \mathfrak{ks}_{\mathfrak{b}(\tau)}(e_3) = [q y_1 y_2].$$

Here  $q \in \mathbb{Q}$  is defined as follows (see Definition 6.7 in [FOOO11a]). Let  $\beta_1 + \beta_2$  be the element of  $H_2(S^2 \times S^2, L(u_0); \mathbb{Z})$  satisfies

$$(\beta_1 + \beta_2) \cap D_i = 1 \quad (i = 1, 2)$$

with Maslov index  $\mu_L(\beta_1 + \beta_2) = 4$  and

$$q := ev_{0*}[\mathcal{M}_{1;1}^{\text{main}}(L(u_0), \beta_1 + \beta_2; e_3)] \cap L(u_0),$$

where we denote by  $\mathcal{M}_{1;1}^{\text{main}}(L(u_0), \beta_1 + \beta_2; e_3)$  the moduli space of genus zero bordered stable maps in class  $\beta_1 + \beta_2$  with one boundary point and one interior point whose image lies in  $D_1 \cap D_2$  (see Section 6 of [FOOO11a] for the precise definition of the moduli space).

The classification theorem of holomorphic disks in [CO06] implies  $q = \pm 1$  immediately.

By comparing  $e_{\epsilon_1, \epsilon_2}^{\tau}$  with  $1_{\epsilon_1, \epsilon_2}^{\tau}$ , we can obtain for  $(\epsilon_1, \epsilon_2) = (\pm 1, \pm 1)$ ,

$$e_{\epsilon_1, \epsilon_2}^{\tau} = \frac{1}{4}(e_0 + \epsilon_1 e^{-\frac{a}{2}} T^{-\frac{1}{2}} \cdot e_1 + \epsilon_2 e^{\frac{a}{2}} T^{-\frac{1}{2}} \cdot e_2 + \epsilon_1 \epsilon_2 q^{-1} T^{-1} \cdot e_3).$$

Since  $a = T^{\frac{1}{2}-\tau}$  and  $0 < \tau < 1/2$ , we obtain  $\mathfrak{v}_T(e_{\epsilon_1, \epsilon_2}^{\tau}) = -1$ . This implies Lemma 5.1.  $\square$

## REFERENCES

- [BEP04] P. Biran, M. Entov and L. Polterovich. “Calabi quasimorphisms for the symplectic ball.” Commun. Contemp. Math., 6.05 (2004): 793-802.
- [Ch00] Yu. V. Chekanov. “Invariant Finsler metrics on the space of Lagrangian embeddings.” Math. Zeitschrift 234.3 (2000): 605-619.
- [CO06] C. H. Cho, Y. G. Oh, “Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds.” Asian J. Math. 10 (2006): 773-814.

- [EP03] M. Entov and L. Polterovich. “*Calabi quasimorphism and quantum homology.*” International Math. Research Notices 30 (2003): 1635-1676.
- [EP06] M. Entov and L. Polterovich. “*Quasi-states and symplectic intersections.*” Comment. Math. Helv. 81 (2006): 75-99.
- [EP09] M. Entov and L. Polterovich. “*Rigid subsets of symplectic manifolds.*” Compositio Mathematica, 145.03 (2009): 773-826.
- [FOOO10] K. Fukaya, Y. G. Oh, H. Ohta, K. Ono. “*Lagrangian Floer theory and mirror symmetry on compact toric manifolds.*” arXiv:1009.1648v2 (2010).
- [FOOO11a] K. Fukaya, Y. G. Oh, H. Ohta, K. Ono. “*Lagrangian Floer theory on compact toric manifolds II: bulk deformations.*” Selecta Math. 17.3 (2011): 609-711.
- [FOOO11b] K. Fukaya, Y. G. Oh, H. Ohta, K. Ono. “*Spectral invariants with bulk quasi-morphisms and Lagrangian Floer theory.*” arXiv:1105.5123 (2011).
- [FOOO12] K. Fukaya, Y. G. Oh, H. Ohta, K. Ono. “*Toric degeneration and nondisplaceable Lagrangian tori in  $S^2 \times S^2$ .*” Int. Math. Research Notices, 2012(13): 2942-2993.
- [Ho90] H. Hofer. “*On the topological properties of symplectic maps.*” Proc. Royal Soc. Edinburgh 115.1-2 (1990): 25-38.
- [Kh09] M. Khanevsky. “*Hofer’s metric on the space of diameters.*” J. Topol. and Anal. 1.04 (2009): 407-416.
- [Oh97] Y. G. Oh. “*Symplectic topology as the geometry of action functional. I.*” J. Diff. Geom. 46.3 (1997): 499-577.
- [Oh05] Y. G. Oh. “*Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds.*” The breadth of symplectic and Poisson geom. Birkhäuser Boston(2005): 525-570.
- [OU13] J. Oakley, M. Usher. “*On certain Lagrangian submanifolds of  $S^2 \times S^2$  and  $\mathbb{C}P^n$ .*” arXiv:1311.5152 (2013)
- [Se14] S. Seyfaddini. “*Unboundedness of the Lagrangian Hofer distance in the Euclidean ball.*” Electron. Res. Announc. Math. Sci. 21 (2014): 1-7.
- [Sc00] M. Schwarz. “*On the action spectrum for closed symplectically aspherical manifolds.*” Pacific Journal of Math. 193.2 (2000): 419-461.
- [Us11] M. Usher. “*Deformed Hamiltonian Floer theory, capacity estimate, and Calabi quasimorphisms.*” Geom. Topol. 15 (2011): 1313?1417.
- [Vi92] C. Viterbo. “*Symplectic topology as the geometry of generating functions.*” Mathematische Annalen 292.1 (1992): 685-710.

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